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Multiple scattering effects in electromagnetic wave propagation through a medium containing precipitation

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Abstract. The general scattering problem of vector electromagnetic waves from a stochastic medium containing particles with random size, orientation and position distributions is formulated and solved in terms of a generalised dyadic average scattering amplitude. The independence approximation used in the solution is verified by considering the results of an exact solution to the scattering from two spheres as a function of their displacement. The specific case of a rain-filled medium and propagation in the centimetre and millimetre wave regions is considered. The general theory is applied in order to derive a more generalised form of the Van de Hulst propagation equation for the coherent field in such a medium. Conditions are then formulated for multiple scattering to be negligible and numerical examples presented covering the frequency range up to 100 GHz which show that indeed the effects of multiple scattering are negligible for the raindrop medium.

1. Introduction

Many electromagnetic wave propagation problems can be modelled by the scattering from an ensemble of particles. Often the scattering from one discrete particle in isolation is soluble but the multiple scattering from the assemblage is very complicated and can be solved exactly only for special cases. A prerequisite to any solution is the physical and geometrical description of the scattering system. In order to reduce the complexity of some propagation problems the state of the system is taken to be deterministic. However, in this work we shall primarily be concerned with the average field and hence coherent scattering from precipitation particles in the troposphere and the statistical nature of their description leads us to formulate the problem in terms of a stochastic model.

The multiple scattering of waves is encountered in many fields and has been the subject of research for many years. Most of the solutions developed are based on the ideas originated by Foldy (1945) and Lax (1951, 1952) and a good survey of the research prior to 1960 is given by Twersky (1960). For deterministically distributed scatterers the problem has been investigated by Twersky (1962a, 1967). Although the deterministic model can be considered as a special case for the precipitation environment, a more general stochastic model is considered essential for meaningful results to be obtained. For the stochastic model the approach and method of solution is different from the deterministic case in which many simplifications can often be made. For a system of randomly distributed scatterers the multiple scattering problem has been investigated by Waterman and Yeh (1961) for scalar waves, and Twersky

(1962b, c) has formulated the problem by using two different approximation techniques. Mathur and Yeh (1964) have used a similar approach to tackle the more complex vector electromagnetic wave case for spherical scatterers. In all of these cases in order to obtain analytic results approximations must be made. In general little attention is paid to the validity of these approximations although Fikioris and Waterman (1964) do consider this problem. A new approach to the problem has been the use of the Monte-Carlo method (Olsen *et al* 1976) but this is very expensive in terms of computer power.

For propagation in rainfall it is usually assumed that multiple scattering effects are negligible (Oguchi 1973). However, the validity of this assumption has received only scant attention in the literature (Ishimara and Lin 1973) for the frequency range 1–100 GHz in which rain/electromagnetic wave interaction is most prominent. In this paper we present a more generalised version of Twersky's theory for vector electromagnetic waves and a stochastic scatterer model which includes random size, orientation and position distributions. The results of an exact solution for multiple scattering between two spheres are presented to justify the main theory and results are presented for the raindrop case which include conditions for multiple scattering to be negligible.

2. General formulation of the multiple scattering problem

2.1. A stochastic model for the multiple scattering medium

Let us define each scatterer by the following parameters:

- $\boldsymbol{r}_i$: the position of the scatterer given as a position vector of a characteristic point of the ith scatterer
- $\hat{\boldsymbol{\alpha}}_i$: the orientation vector of the ith scatterer
- a_i: an equivolumetric spherical radius defining the size of the scatterer.

In addition the shape of the scatterer should be specified. For the raindrop case the shape is spheroidal and the general description given here is appropriate. Let the number of scatterers forming the ensemble be $N \gg 1$; then in order to obtain a stochastic description of the system we must find a composite probability distribution

$$p(\boldsymbol{r}_1, \hat{\boldsymbol{\alpha}}_1, a_1; \ldots; \boldsymbol{r}_N, \hat{\boldsymbol{\alpha}}_N, a_N). \tag{1}$$

In order to simplify the analysis we will assume all of the stochastic variables to be statistically independent, i.e.

$$p(\boldsymbol{r}_1, \hat{\boldsymbol{\alpha}}_1, a_1; \ldots; \boldsymbol{r}_N, \hat{\boldsymbol{\alpha}}_N, a_N) = \prod_{i=1}^{N} p(\boldsymbol{r}_i) p(\hat{\boldsymbol{\alpha}}_i) p(a_i) \tag{2}$$

where $p(\boldsymbol{r}_i)$, $p(\hat{\boldsymbol{\alpha}}_i)$ and $p(a_i)$ are the probability distributions of the random variable's position, orientation and size. In addition we assume that there is no interactive force between the particles such that the position of the particles is a uniform distribution in three-dimensional space. Accordingly,

$$p(\boldsymbol{r}_i) = \rho/N \qquad \text{for } i = 1, 2 \ldots, N \tag{3}$$

where ρ is the number of particles per unit volume. The scatterers are confined inside a slab of thickness d and the complete geometry is shown in figure 1. Let us now investigate the electromagnetic problem.

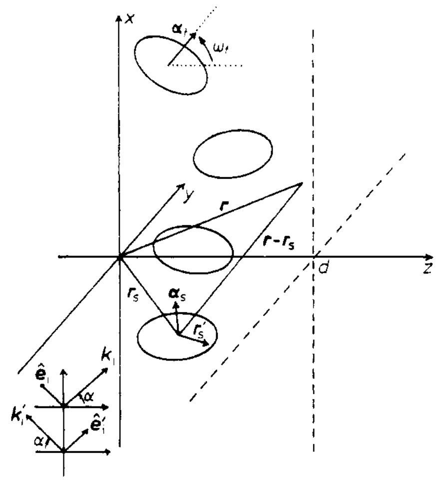

Figure 1. Multiple scattering geometry.

For an incident wavevector $\boldsymbol{k}_i$ the 'dyadic electric field $\tilde{\boldsymbol{E}}(r)$' will satisfy an integral equation of the form

$$\tilde{\boldsymbol{E}}(r)=\tilde{\boldsymbol{E}}_0(r)+\sum_{s=1}^{N}\int_{V_s}\mathrm{d}\boldsymbol{r}'_s\,\tilde{\boldsymbol{\Gamma}}(|\boldsymbol{r}-\boldsymbol{r}_s-\boldsymbol{r}'_s|)\cdot\tilde{\boldsymbol{E}}(\boldsymbol{r}_s+\boldsymbol{r}'_s)=\tilde{\boldsymbol{E}}_0(\boldsymbol{r})+\tilde{\boldsymbol{\mathscr{E}}}(\boldsymbol{r}) \tag{4}$$

where $\tilde{\boldsymbol{E}}_0(\boldsymbol{r})=(\tilde{\mathbb{1}}-\hat{\boldsymbol{k}}_i\hat{\boldsymbol{k}}_i)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r}}$ is the incident dyadic field, i.e. $\tilde{\boldsymbol{E}}_0(\boldsymbol{r})\cdot\hat{\boldsymbol{e}}_i=\hat{\boldsymbol{e}}_i\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{r}}$; $\hat{\boldsymbol{e}}_i$ being a unit incident field polarisation vector. The dyadic Green function is given:

$$\tilde{\boldsymbol{\Gamma}}(|\boldsymbol{r}-\boldsymbol{r}'|)=\frac{k_0^2(n^2-1)}{4\pi}(\tilde{\mathbb{1}}+k_0^{-2}\boldsymbol{\nabla}'\boldsymbol{\nabla}')\frac{\mathrm{e}^{\mathrm{i}k_0|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{5}$$

$k_0=|\boldsymbol{k}_i|$ and n is the refractive index of the scatterers (in this case we assume that all scatterers have the same refractive index), $\boldsymbol{r}'_s$ is the integration variable which is taken with respect to the origin $\boldsymbol{r}_s$ of the sth scatterer (i.e. the local coordinate system of the sth particle) and $\tilde{\mathbb{1}}$ is the unit dyadic.

2.2. *The average scattered field* $\langle\tilde{\boldsymbol{\mathscr{E}}}(\boldsymbol{r})\rangle$

The ensemble average of the random scattered field is given by

$$\langle\tilde{\boldsymbol{\mathscr{E}}}(\boldsymbol{r})\rangle=\int p(\boldsymbol{r}_1,\hat{\boldsymbol{\alpha}}_1,a_1;\ldots;\boldsymbol{r}_N,\hat{\boldsymbol{\alpha}}_N,a_N)\tilde{\boldsymbol{\mathscr{E}}}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\hat{\boldsymbol{\alpha}}_1\,\mathrm{d}a_1\ldots\mathrm{d}\boldsymbol{r}_N\,\mathrm{d}\hat{\boldsymbol{\alpha}}_N\,\mathrm{d}a_N$$

$$=\sum_{s=1}^{N}\int p(\boldsymbol{r}_s)p(\hat{\boldsymbol{\alpha}}_s)p(a_s)\,\mathrm{d}\boldsymbol{r}_s\,\mathrm{d}\tilde{\boldsymbol{\alpha}}_s\,\mathrm{d}a_s\int_{V_s}\mathrm{d}\boldsymbol{r}'_s\,\tilde{\boldsymbol{\Gamma}}(|\boldsymbol{r}-\boldsymbol{r}_s-\boldsymbol{r}'_s|)\cdot\langle\tilde{\boldsymbol{E}}(\boldsymbol{r}_s+\boldsymbol{r}'_s)\rangle_s \tag{6}$$

where $\langle\,\rangle_s$ means that the ensemble average is taken by keeping the sth scatterer constant in position, size and orientation. Substituting equation (3) into (6) and assuming $\hat{\boldsymbol{\alpha}}_s$, a_s have the same statistics for all particles

$$\langle \tilde{\mathscr{E}}(r)\rangle = \rho \int \mathrm{d}\boldsymbol{r}_s \int p(\hat{\boldsymbol{\alpha}}_s)p(a_s)\,\mathrm{d}\hat{\boldsymbol{\alpha}}_s\,\mathrm{d}a_s \int_{V_s} \mathrm{d}\boldsymbol{r}'_s\,\tilde{\boldsymbol{\Gamma}}(|\boldsymbol{r}-\boldsymbol{r}_s-\boldsymbol{r}'_s|)\,.\,\langle \tilde{\boldsymbol{E}}(\boldsymbol{r}_s+\boldsymbol{r}'_s)\rangle_s. \tag{7}$$

A translation of the sth scatterer parallel to x, y plane so that its centre $\boldsymbol{r}_s$ falls on the z axis at a point z_s, according to equation (4) involves a phase change of

$$\langle \tilde{\boldsymbol{E}}(\boldsymbol{r}_s+\boldsymbol{r}'_s)\rangle_s = \mathrm{e}^{\mathrm{i}k_0 x_s \sin\alpha}\,\langle \tilde{\boldsymbol{E}}(z_s+\boldsymbol{r}'_s)\rangle_s. \tag{8}$$

By using the complex wave representation for the Green function, namely;

$$\frac{\mathrm{e}^{\mathrm{i}k_0|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{\mathrm{i}k_0}{2\pi}\int_0^{2\pi}\mathrm{d}\beta \int_0^{\frac{1}{2}\pi-\mathrm{i}\infty} \sin\zeta\,\mathrm{d}\zeta\,\exp[\![\mathrm{i}k_0\{|z-z'|\cos\zeta$$
$$+\sin\zeta[(x-x')\cos\beta+(y-y')\sin\beta]\}]\!] \tag{9}$$

together with equations (7) and (8) and performing the integrations we obtain,

$$\langle \tilde{\mathscr{E}}(\boldsymbol{r})\rangle = \frac{\mathrm{i}k_0\rho(n^2-1)}{2\cos\alpha}\int_0^d \mathrm{d}z_s \int p(\hat{\boldsymbol{\alpha}}_s)p(a_s)\,\mathrm{d}a_s\,\mathrm{d}\hat{\boldsymbol{\alpha}}_s \int_{V_s}\mathrm{d}\boldsymbol{r}'_s(\tilde{\mathbb{1}}+k_0^{-2}\boldsymbol{\nabla}'\boldsymbol{\nabla}')$$
$$\times \exp\{\mathrm{i}k_0[|z-z_s-z'_s|\cos\alpha+(x-x'_s)\sin\alpha]\}\,.\,\langle \tilde{\boldsymbol{E}}(z_s+\boldsymbol{r}'_s)\rangle_s \tag{10}$$

where α is defined as the angle between the z axis and $\hat{\boldsymbol{k}}_\mathrm{i}(=(\sin\alpha, 0, \cos\alpha))$. For a point $\boldsymbol{r}$ inside the scattering medium we can write (10) as

$$\langle \tilde{\mathscr{E}}(\boldsymbol{r})\rangle = \frac{2\pi\mathrm{i}\rho}{\gamma}\left(\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{i}\cdot\boldsymbol{r}}\int_0^z \mathrm{d}z_s\langle \tilde{\boldsymbol{F}}(\boldsymbol{k}_\mathrm{i}|z_s)\rangle\,\mathrm{e}^{-\mathrm{i}\gamma z_s}+\mathrm{e}^{\mathrm{i}\boldsymbol{k}'_\mathrm{i}\cdot\boldsymbol{r}}\int_z^d \mathrm{d}z_s\,\langle \tilde{\boldsymbol{F}}(\boldsymbol{k}'_\mathrm{i}|z_s)\rangle\,\mathrm{e}^{\mathrm{i}\gamma z_s}\right) \tag{11}$$

where $\gamma = k_0\cos\alpha$; $\boldsymbol{k}'_\mathrm{i} = k_0(\sin\alpha, 0, -\cos\alpha)$ and

$$\langle \tilde{\boldsymbol{F}}(\boldsymbol{k}_\mathrm{i}, z_s) = \frac{k_0^2(n^2-1)}{4\pi}\int p(\hat{\boldsymbol{\alpha}}_s)p(a_s)\,\mathrm{d}\hat{\boldsymbol{\alpha}}_s\,\mathrm{d}a_s \int_{V_s}\mathrm{d}\boldsymbol{r}'_s(\tilde{\mathbb{1}}+k_0^{-2}\boldsymbol{\nabla}'\boldsymbol{\nabla}')$$
$$.\,\mathrm{e}^{-\mathrm{i}k_0(x'_s\sin\alpha+z'_s\cos\alpha)}\,\langle \tilde{\boldsymbol{E}}(z_s+\boldsymbol{r}'_s)\rangle_s \tag{12}$$

the latter being a 'generalised scattering amplitude'.

Now defining $\tilde{\boldsymbol{E}}_+(0, z)$, $\tilde{\boldsymbol{E}}_-(z, d)$ as

$$\tilde{\boldsymbol{E}}_+(0, z) = \mathrm{e}^{\mathrm{i}\gamma z}\left((\tilde{\mathbb{1}}-\hat{\boldsymbol{k}}_\mathrm{i}\hat{\boldsymbol{k}}_\mathrm{i})+\frac{2\pi\rho\mathrm{i}}{\gamma}\int_0^z \mathrm{d}z_s\,\mathrm{e}^{-\mathrm{i}\gamma z_s}\langle \tilde{\boldsymbol{F}}(\boldsymbol{k}_\mathrm{i}|z_s)\rangle\right) \tag{13}$$

$$\tilde{\boldsymbol{E}}_-(z, d) = \mathrm{e}^{-\mathrm{i}\gamma z}\frac{2\pi\rho\mathrm{i}}{\gamma}\int_z^d \mathrm{d}z_s\,\mathrm{e}^{\mathrm{i}\gamma z_s}\langle \tilde{\boldsymbol{F}}(\boldsymbol{k}'_\mathrm{i}|z_s)\rangle \tag{14}$$

we can write the average field as follows

$$\langle \tilde{\boldsymbol{E}}(\boldsymbol{r})\rangle = (\tilde{\boldsymbol{E}}_+(0, z)+\tilde{\boldsymbol{E}}_-(z, d))\,\mathrm{e}^{\mathrm{i}k_0 x\sin\alpha}. \tag{15}$$

2.3. *Evaluation of the average scattered field*

Equation (11) describes the average value of the scattered field in terms of an unknown generalised scattering amplitude which in turn is related to the unknown average scattered field $\langle \tilde{\boldsymbol{E}}(z_s+\boldsymbol{r}'_s)\rangle_s$.

By using the conditional probability rule we can write

$$\langle\tilde{\boldsymbol{E}}(\boldsymbol{z}_s+\boldsymbol{r}'_s)\rangle_s=\tilde{\boldsymbol{E}}_0(\boldsymbol{z}_s+\boldsymbol{r}'_s)+\int_{V^s}\mathrm{d}\boldsymbol{r}''_s\,\tilde{\boldsymbol{\Gamma}}(|\boldsymbol{r}'_s-\boldsymbol{r}''_s|)\cdot\langle\tilde{\boldsymbol{E}}(\boldsymbol{z}_s+\boldsymbol{r}''_s)\rangle_s$$
$$+\sum_{\lambda\neq s}\left(\int \mathrm{d}\boldsymbol{r}_\lambda\,\mathrm{d}\hat{\boldsymbol{\alpha}}_\lambda\,\mathrm{d}a_\lambda\,p(\boldsymbol{r}_\lambda/\boldsymbol{r}_s)p(a_\lambda)p(\hat{\boldsymbol{\alpha}}_\lambda)\right.$$
$$\left.\times\int_{V_\lambda}\tilde{\boldsymbol{\Gamma}}(|\boldsymbol{z}_s+\boldsymbol{r}'_s-\boldsymbol{r}_\lambda-\boldsymbol{r}'_\lambda|)\cdot\langle\tilde{\boldsymbol{E}}(\boldsymbol{r}_\lambda+\boldsymbol{r}'_\lambda)\rangle_{\lambda,s}\,\mathrm{d}\boldsymbol{r}'_\lambda\right) \tag{16}$$

where the conditional probability is given as

$$p(\boldsymbol{r}_\lambda/\boldsymbol{r}_s)=p(\boldsymbol{r}_\lambda)=\rho/N \tag{17}$$

and $\langle\ \rangle_{\lambda,s}$ indicates that the ensemble average is taken keeping the λ and sth scatterer's random variables fixed. The recursive procedure in (16) can be applied successively, increasing the number of fixed scatterers. In order to achieve a solution this chain process should be broken at some stage. In order to accomplish this it is necessary to assume that the scatterer system is such that we can impose the approximation

$$\langle\tilde{\boldsymbol{E}}(\boldsymbol{r}_s+\boldsymbol{r}'_s)\rangle_{s,\lambda}\approx\langle\tilde{\boldsymbol{E}}(\boldsymbol{r}_s+\boldsymbol{r}'_s)\rangle_s. \tag{18}$$

This implies that the average value of the field inside the sth scatterer is unaffected by a change of state of the λth scatterer. The validity of this approximation is considered in the next subsection.

Substituting equations (17), (18) into (16) and assuming that $N\gg 1$ we obtain

$$\langle\tilde{\boldsymbol{E}}(\boldsymbol{z}_s+\boldsymbol{r}'_s)\rangle_s=\langle\tilde{\boldsymbol{E}}(\boldsymbol{z}_s+\boldsymbol{r}_s)\rangle+\int_{V^s}\mathrm{d}\boldsymbol{r}''_s\,\tilde{\boldsymbol{\Gamma}}(|\boldsymbol{r}'_s-\boldsymbol{r}''_s|)\cdot\langle\tilde{\boldsymbol{E}}(\boldsymbol{z}_s+\boldsymbol{r}''_s)\rangle_s. \tag{19}$$

Now it has already been shown that $\langle\tilde{\boldsymbol{E}}(\boldsymbol{z}_s+\boldsymbol{r}'_s)\rangle$, given by equation (15), consists of the superposition of two plane waves. Equation (19) states that the average field $\langle\tilde{\boldsymbol{E}}(\boldsymbol{z}_s+\boldsymbol{r}'_s)\rangle_s$, keeping the sth scatterer fixed, is the solution of the scattering problem when the incident wave is given by equation (15).

It should be noticed that the generalised scattering amplitude defined in equation (12) corresponds to this scattering process. Accordingly we can now write the generalised scattering amplitude as the superposition of the individual scattering amplitudes multiplied by the appropriate exciting fields, namely:

$$\langle\tilde{\boldsymbol{F}}(\boldsymbol{k}_t|z)\rangle=\int p(\hat{\alpha})p(a)\,\mathrm{d}a\,\mathrm{d}\hat{\boldsymbol{\alpha}}\left(\tilde{\boldsymbol{f}}(\boldsymbol{k}_t|\boldsymbol{k}_\mathrm{i},\hat{\alpha},a)\cdot\tilde{\boldsymbol{E}}_+(0,z)+\tilde{\boldsymbol{f}}(\boldsymbol{k}_t|\boldsymbol{k}'_\mathrm{i},\hat{\boldsymbol{\alpha}},a)\cdot\tilde{\boldsymbol{E}}_-(z,d)\right) \tag{20}$$

where $\boldsymbol{k}_t$ is the wavevector in the observation direction and $\tilde{\boldsymbol{f}}(\boldsymbol{k}_t|\boldsymbol{k}_\mathrm{i},\hat{\boldsymbol{\alpha}},a)$ is the individual dyadic scattering amplitude for a scatterer with orientation $\hat{\boldsymbol{\alpha}}$ and size a.

Defining,

$$\langle\tilde{\boldsymbol{f}}(\boldsymbol{k}_t|\boldsymbol{k}_\mathrm{i})\rangle=\int p(\hat{\boldsymbol{\alpha}})p(a)\,\mathrm{d}a\,\mathrm{d}\hat{\boldsymbol{\alpha}}\,\tilde{\boldsymbol{f}}(\boldsymbol{k}_t|\boldsymbol{k}_\mathrm{i},\hat{\boldsymbol{\alpha}},a) \tag{21}$$

equation (20) can be written as,

$$\langle\tilde{\boldsymbol{F}}(\boldsymbol{k}_t|z)\rangle=\langle\tilde{\boldsymbol{f}}(\boldsymbol{k}_t|\boldsymbol{k}_\mathrm{i})\rangle\cdot\tilde{\boldsymbol{E}}_+(0,z)+\langle\boldsymbol{f}(\boldsymbol{k}_t|\boldsymbol{k}'_\mathrm{i})\rangle\cdot\tilde{\boldsymbol{E}}_-(z,d). \tag{22}$$

Substituting the values of $\langle \tilde{\boldsymbol{F}}(\boldsymbol{k}_t|\boldsymbol{z})\rangle$ into equations (13), (14) and differentiating twice we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}\tilde{\boldsymbol{E}}_+(0,z)\\ \tilde{\boldsymbol{E}}_-(z,d)\end{pmatrix}=\begin{pmatrix}\mathrm{i}\gamma & 0\\ 0 & -\mathrm{i}\gamma\end{pmatrix}\cdot\begin{pmatrix}\tilde{\boldsymbol{E}}_+(0,z)\\ \tilde{\boldsymbol{E}}_-(z,d)\end{pmatrix}+\begin{pmatrix}\tilde{\boldsymbol{S}}_{11} & \tilde{\boldsymbol{S}}_{12}\\ -\tilde{\boldsymbol{S}}_{21} & -\tilde{\boldsymbol{S}}_{22}\end{pmatrix}\cdot\begin{pmatrix}\tilde{\boldsymbol{E}}_+(0,z)\\ \tilde{\boldsymbol{E}}_-(z,d)\end{pmatrix} \tag{23}$$

where

$$\begin{aligned}\tilde{\boldsymbol{S}}_{11}&=c\langle\tilde{\boldsymbol{f}}(\boldsymbol{k}_i|\boldsymbol{k}_i)\rangle & \tilde{\boldsymbol{S}}_{12}&=c\langle\tilde{\boldsymbol{f}}(\boldsymbol{k}_i|\boldsymbol{k}_i')\rangle\\ \tilde{\boldsymbol{S}}_{21}&=c\langle\tilde{\boldsymbol{f}}(\boldsymbol{k}_i'|\boldsymbol{k}_i)\rangle & \tilde{\boldsymbol{S}}_{22}&=c\langle\tilde{\boldsymbol{f}}(\boldsymbol{k}_i'|\boldsymbol{k}_i')\rangle\end{aligned} \tag{24}$$

where $c=2\pi\mathrm{i}\rho/\gamma$.

Equation (23) describes the propagation inside the multiple scattering medium, assuming that we know the individual scattering amplitudes of the scatterers therein.

2.4. *Validity of the independence approximation*

The independence approximation given in equation (18) is the basis for the solution of the ensemble scattering problem given in the previous subsection. Equation (18) states that averaging over all possible values for $(N-2)$ scatterers the field inside the sth scatterer is not affected by the state of the λth scatterer. The latter will certainly be true if multiple scattering between S and λ is weak. To investigate the latter even for the raindrop situation over a reasonable size and frequency range is very difficult. We have taken the simplest model of two neighbouring spheres with the complex refractive index of water and investigated scattering of scalar waves from them. This precise problem has not previously been evaluated although the scattering of a plane electromagnetic wave by two spherical particles has received considerable attention in the the literature (e.g. see Twersky 1967). We have used an extension of the method of multipole expansion to obtain an exact solution for the scattered fields from the spheres making use of the translational addition theorem for vector spherical wave-functions. The detailed mathematical derivation is not included herein but is to be found in Uzunoglu (1976). In order to demonstrate the validity of equation (18) and to show the effect of multiple scattering, two specific cases have been evaluated, one in the millimetre wave range and the other for optical frequencies. In both cases besides the exact total scattering amplitude two approximations have been evaluated. The first is the independent scattering amplitude which assumes that the individual scattering amplitudes sum up in the forward direction and the second an intermediate approximation which assumes a far-field influence of one scatterer on its neighbour. The two specific results are shown in figure 2 where the extinction and scattering cross section respectively are shown for the three cases as a function of the displacement b between equal sphere centres.

Figure 2(a) shows a large refractive index case ($\eta=5{\cdot}8+2{\cdot}1\mathrm{i}$, $k_0a_1=k_0a_2=0{\cdot}4$) with endfire incidence. This corresponds to a medium size raindrop at millimetre wave frequencies. For the forward scattering amplitude, the difference in the extinction cross section given by the two approximations and the exact solution is less than 15% even for small values of k_0b. As k_0b increases the total extinction cross section of the exact and approximate solution converge to the same values. The exact solution is seen to oscillate with period $2k_0b/\pi$.

A second example is shown in figure 2(b) for a pure-real refractive index $\eta=1{\cdot}333$, $k_0a_1=k_0a_2=2{\cdot}0$. This corresponds to a small size raindrop at optical

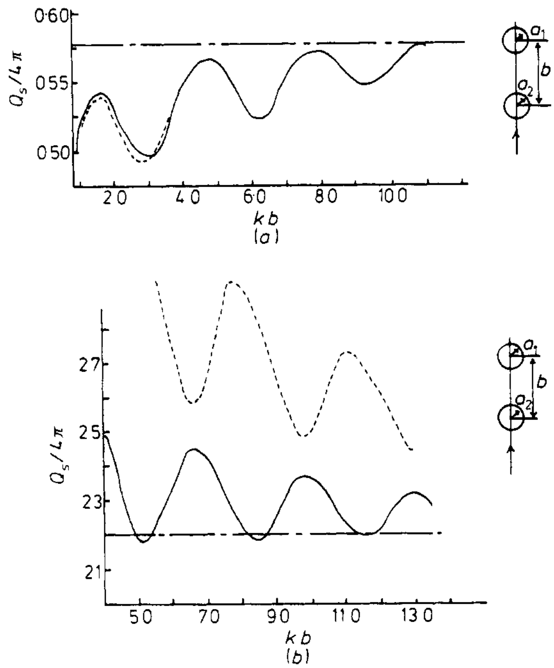

Figure 2. (*a*) Extinction cross section dependence for two spherical scatterers, $n_0 = 5{\cdot}8 + i2{\cdot}1$, $k_0a_1 = k_0a_2 = 0{\cdot}4$. (*b*) Scattering cross section dependence for two spherical scatterers, $n_0 = 1{\cdot}33333$, $k_0a_1 = k_0a_2 = 2{\cdot}0$. For both (*a*) and (*b*), ——, exact solution; — — —, independent scatterer approximation; - - - -, far-field approximation.

frequencies. Multiple scattering phenomena are seen to be strong for small values of k_0b and the far-field approximation breaks down and cannot be used as a basis for evaluation. Although both of the approximations converge to the independent scatterer cross section the rate of convergence is much slower than in figure 2(*a*). The essential condition for multiple scattering to be negligible is that $k_0b \gg 1$. For a given k_0a there is for the raindrop case a value of $k_0b \gg 1$ for which the independence approximation is valid.

The two neighbouring spheres problem has been solved for scalar waves but as the basic equations are similar in the electromagnetic case the same qualitative conclusions regarding the validity of the independence relationship apply to both.

3. Propagation of centimetre and millimetre waves through a region containing spheroidal raindrops

3.1. A description of the propagation medium

In order to solve for propagation through a stochastic medium we need to use the general equations (23) and (24). However, before doing this let us make some simplifying assumptions. Firstly, let the orientation of scatterers be peaked around $\hat{\boldsymbol{\alpha}}_0$. (There is some experimental evidence to justify this.) That is:

$$p(\hat{\boldsymbol{\alpha}}) \simeq \delta(\hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\alpha}}_0) \tag{25}$$

and

$$\langle \tilde{f}(\boldsymbol{k}_t|\boldsymbol{k}_i)\rangle = \int_0^{+\infty} p(a)\,\mathrm{d}a\,\tilde{f}(\boldsymbol{k}_t|\boldsymbol{k}_i, \hat{\boldsymbol{\alpha}}_0, a)$$

where the raindrop size distribution $p(a)\,\mathrm{d}a$ has been investigated by many authors in the literature, perhaps most notably Marshall and Palmer (1948) and Laws and Parsons (1943).

Let us assume that our basic polarisation vectors are chosen as

$$\hat{\boldsymbol{e}}_1 \parallel \hat{\boldsymbol{k}}_i \times (\hat{\boldsymbol{\alpha}}_0 \times \hat{\boldsymbol{k}}_i); \qquad |\hat{\boldsymbol{e}}_1| = 1 \tag{26}$$

$$\hat{\boldsymbol{e}}_2 \parallel \hat{\boldsymbol{\alpha}}_0 \times \hat{\boldsymbol{k}}_i; \qquad |\hat{\boldsymbol{e}}_2| = 1. \tag{27}$$

From previous work Uzunoglu *et al* (1978) the dyadic scattering amplitudes for the forward and backward directions for a spheroid have been calculated from

$$\tilde{f}(\boldsymbol{k}_t|\boldsymbol{k}_i, \hat{\boldsymbol{\alpha}}_0, a) = f_1(\boldsymbol{k}_t|\boldsymbol{k}_i, \hat{\boldsymbol{\alpha}}_0, a)\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_1 + f_2(\boldsymbol{k}_t|\boldsymbol{k}_i, \hat{\boldsymbol{\alpha}}_0, a)\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_2$$

for the cases where $\boldsymbol{k}_t = \pm\boldsymbol{k}_i$.

We may now define the averaged quantities,

$$\langle f_j(\boldsymbol{k}_t|\boldsymbol{k}_i)\rangle = \int p(a)\,\mathrm{d}a\,f_j(\boldsymbol{k}_t|\boldsymbol{k}_i, \hat{\boldsymbol{\alpha}}_0, a) \qquad \text{for } j = 1, 2 \text{ and } \boldsymbol{k}_t = \pm\boldsymbol{k}_i$$

and

$$\left.\begin{aligned} f_j(0) &= \langle f_j(\boldsymbol{k}_i|\boldsymbol{k}_i)\rangle \\ f_j(\pi) &= \langle f_j(-\boldsymbol{k}_i|\boldsymbol{k}_i)\rangle \end{aligned}\right\} \qquad \text{for } j = 1, 2.$$

Now using the above in equations (23) and (24) the wavevectors k_j corresponding to the two possible polarisations may be evaluated as

$$k_j = k_0\left[\left(1 + \frac{2\pi\rho}{k_0^2}(f_j(0) - f_j(\pi))\right)\left(1 + \frac{2\pi\rho}{k_0^2}(f_j(0) + f_j(\pi))\right)\right]^{1/2} \qquad \text{for } j = 1, 2. \tag{28}$$

3.2. Multiple scattering phenomena

Equation (28) is a generalised version of the Van de Hulst (1957) scattering equation for a stochastic propagation medium. It is valid providing the validity of equation (18), the independence approximation, is established. The $j = 1, 2$ subscripts in the above correspond to the cases of vertical and horizontal polarisation. It will be noticed that the generalised equation (28) reduces to the familiar Van de Hulst single scattering equation ($k_j = k_0 + (2\pi\rho/k_0)f_j(0)$) if the following inequality holds:

$$\left|\left(1 + \frac{2\pi\rho}{k_0^2}f_j(0)\right)\right| > \frac{2\pi\rho}{k_0^2}|f_j(\pi)| \qquad \text{for } j = 1, 2. \tag{29}$$

Thus it may be concluded that if the above is true and equation (18) is valid then the Van de Hulst approximation can be used in order to describe propagation in a random rain-filled medium.

Although the above inequality is related to the entire properties of the rain-filled medium the most important factor is the distance between the raindrop centres. For example, for very high rainfall rates, e.g. 150 mm/h, the mean distance between drops is $b \simeq 10$ cm and thus for heavy rainfall frequencies, greater than 10 GHz, $k_0 b \geqslant 21$. From the analysis of the two spherical scatterer problems it can be seen that for

$k_0 b > 10$ the interaction between scatterers is negligible. Thus for 4–100 GHz we can safely assume that the first condition (equation (18)) is valid. The satisfaction of this condition comes from the 'sparse distribution' of the rain-filled medium. The second condition (equation (29)) may be checked if the averaged forward and backward scattering amplitudes are known. In a previous work (Uzunoglu *et al* 1978) we have evaluated these quantities at 4, 6, 11, 14, 20 and 30 GHz for spheroidal raindrops and using these values it will be seen that the second condition is valid up to 30 GHz. For higher frequencies we have used Mie's spherical scattering theory to obtain averaged forward and backward scattering amplitudes and again have checked that the second condition is valid to 100 GHz.

It may be expected that for thunderstorm type rain the drop size distribution may be peaked at larger drop sizes and thus multiple scattering may be more likely. Let us test this case by assuming that the rain consists only of drops of $a = 0{\cdot}25$ cm; for an incident wave frequency of 100 GHz using Mie theory:

$$f(\boldsymbol{k}_i|\boldsymbol{k}_i, 0{\cdot}25) = -0{\cdot}032 + \mathrm{i}0{\cdot}846$$

$$f(-\boldsymbol{k}_i|\boldsymbol{k}_i, 0{\cdot}25) = 0{\cdot}039 - \mathrm{i}0{\cdot}0561$$

and assuming $p(a)\,\mathrm{d}a = \delta(a - 0{\cdot}25)\,\mathrm{d}a$.

For a rain rate of 150 mm/h using the Marshall–Palmer distribution $\rho = 5{\cdot}58 \times 10^{-3}\ \mathrm{cm}^{-3}$ and $k_0 = 20{\cdot}94$, it will be seen that equation (28) is again valid.

4. Conclusions

We have formulated the general scattering problem (including multiple scattering) for vector electromagnetic waves from a stochastic propagation medium containing scatterers of random size, orientation, shape and position distributions. Thus multiple scattering phenomena may be described in a closed form if an 'independence approximation' is valid. This independence approximation is verified by considering two neighbouring spheres scattering. The particular case of propagation of centimetre and millimetre waves through a rain-filled medium is investigated in detail. A generalised form of the propagation coefficient for such a medium is calculated and shown to reduce to the familiar Van de Hulst single scattering formula in its simplest form. Conditions for multiple scattering to be negligible and for the Van de Hulst equation to be applicable are derived. The latter have been investigated numerically for electromagnetic wave propagation in a rain-filled medium for the frequency range 4–100 GHz. It has been shown that multiple scattering in such a medium will be negligible over this frequency range.

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